Capacity Bounds on MIMO Relay Channel With Covariance Feedback at the Transmitters

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***Abstract*—In this paper, the source and relay transmit covariance matrices are jointly optimized for a fading multiple-antenna relay channel when the transmitters only have partial channel state information (CSI) in the form of covariance feedback. For both full-duplex and half-duplex transmissions, we evaluate lower and upper bounds on the ergodic channel capacity. These bounds require a joint optimization over the source and relay transmit covariance matrices. The methods utilized in the previous literature cannot handle this joint optimization over the transmit covariance matrices for the system model considered in this paper. Therefore, we utilize matrix differential calculus and propose iterative algorithms that find the transmit covariance matrices to solve the joint optimization problem. In this method, there is no need to specify first the eigenvectors of the transmit covariance matrices. The algorithm updates both the eigenvectors and the eigenvalues at each iteration. Through simulations, we observe that lower and upper bounds are close to each other. However, the distance between the lower and upper bounds depends on the channel conditions. If the mutual information on the source-to-relay channel and the broadcast channel get closer to each other, the bounds on capacity also get closer.**

***Index Terms*—Covariance feedback, full duplex, half-duplex, multiple-input–multiple-output (MIMO) relay, optimum power allocation, partial channel state information (CSI).**

I. Introduction

UTILIZING multiple antennas at the transmit and receive terminals of wireless communication systems has been shown to increase spectral efficiency [1]. In addition, applying cooperative strategies such as adding a relay node to the system can further increase capacity [2]. On the other hand, the exact description of the multiple-input–multiple-output (MIMO) relay channel capacity is still an open problem. Several achievable schemes, such as decode-and-forward (DF), amplify-and-forward, and compress-and-forward schemes, can be used as lower bounds to the capacity of MIMO relay channels, whereas the cut-set theorem provides a valid upper bound.

For single-antenna fading relay channels, capacity bounds and power allocations are given in [3] for both full-duplex and half-duplex transmissions, where perfect channel state information (CSI) is available everywhere. A similar setting with individual power constraints at the source and the relay is considered in [4], where a max–min type of solution is also introduced. In [5], MIMO relay channels with different fading assumptions are discussed when only the receivers have the perfect CSI. For full-duplex, fading MIMO relay channels, a capacity upper bound, and a DF achievable rate are found in [6], where only the receivers know the perfect CSI, and the transmitters do not know the channel.

A more practical channel model, for which the receivers have the perfect CSI and the transmitters have partial CSI, was utilized for point-to-point MIMO and MIMO multiple-access channels (MACs) in [7]–[9]. In both of these channels, it is possible to find the eigenvectors of the transmit covariance matrices in closed form, and solve a reduced optimization problem over the eigenvalues of the transmit covariance matrices, using an iterative algorithm [8], [9]. However, in relay channels, it is not always possible to find a closed-form expression for the eigenvectors of the transmit covariance matrices. One solution that we offered to this problem was to choose the eigenvectors of the transmit covariance matrices similar to point-to-point channels [10]. However, this choice is clearly suboptimal. Therefore, here, we propose a new method for solving the transmit covariance matrices directly (i.e., without needing to find the eigenvectors first).

In this method, matrix differential calculus [11] is extremely functional since it offers a new way for optimizing rate expressions by taking derivatives of scalar functions with respect to matrix variables (transmit covariance matrices). This eliminates the need for calculating cumbersome partial differentials that need to be taken with respect to the eigenvalues of matrix variables. By using matrix differential calculus, the resulting iterative algorithm updates the entire matrix at once at each iteration.

In this paper, we consider both full-duplex and half-duplex MIMO relay channels where the transmitters have partial CSI in the form of covariance feedback. The source and relay terminals have individual power constraints. We evaluate the DF lower bound and the cut-set upper bound on the channel capacity that are given in terms of max–min-type optimization problems over the source and relay transmit covariance matrices. The main contribution of this paper is to find the transmit covariance matrices that satisfy the lower and upper bound optimization problems. We solve these joint optimization problems using techniques from [4] as well as by using matrix differential calculus [11]. The solutions to the optimization problems are in terms of iterative algorithms that find the transmit covariance matrices directly (i.e., without first finding the eigenvectors

Manuscript received September 3, 2012; revised December 4, 2012; accepted December 13, 2012. Date of publication January 3, 2013; date of current version June 12, 2013. This work was supported by the Scientific and Technological Research Council of Turkey under Grant EEEAG 109E048. The review of this paper was coordinated by Prof. H. F. Lu.

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Digital Object Identifier 10.1109/TVT.2012.2237423

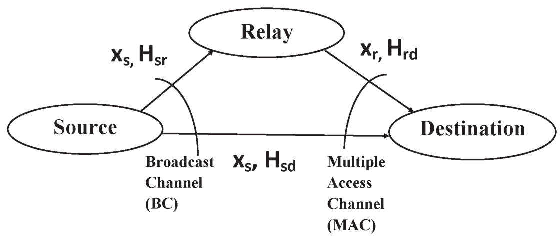

Fig. 1. MIMO relay channel.

and then calculating the eigenvalues). Through simulations, we show that the proposed algorithms converge, regardless of the initial points. Moreover, we observe that, for certain channel covariance matrix settings, lower and upper bounds meet to give the exact capacity.

The following notation is adopted throughout this paper: upper (lower) boldface letters are used to denote matrices (column vectors). Superscript $(\cdot)^\dagger$ stand for the conjugate-transpose operation. We employ $E[\cdot]$ to denote expectation with respect to all random variables within the brackets. Operators $\mathrm{tr}(\cdot)$ and $|\cdot|$ represent the matrix trace and determinant, respectively.

II. System Model

Consider a relay channel with source, relay, and destination terminals (see Fig. 1), where the channel between a transmitter and a receiver is represented by random matrix $\mathbf{H}_{xy}$, whose elements are complex Gaussian random variables resulting in a Rayleigh fading channel. Dimensions of the channel matrix are the number of receive antennas times the number of transmit antennas. In the case that the receiver has the perfect CSI and the transmitter has only statistical knowledge of channel in terms of covariance feedback, there is a correlation between the signals transmitted by or received at different antenna elements. This channel model is defined as [12]

$$\mathbf{H}_{xy} = \mathbf{\Phi}_{xy}^{1/2} \mathbf{Z}_{xy} \mathbf{\Sigma}_{xy}^{1/2} \tag{1}$$

where subscript xy refers to either sr (source to relay), sd (source to destination), or rd (relay to destination); $\mathbf{Z}_{xy}$ is a zero-mean identity covariance random channel matrix; $\mathbf{\Sigma}_{xy}$ is the correlation matrix between the signals transmitted from the antennas on the transmitter; and $\mathbf{\Phi}_{xy}$ is the correlation matrix between the signals received at the antennas on the receiver. Similar to [9], in this paper, we will assume that the receivers do not have any physical restrictions; therefore, there is sufficient spacing between the antenna elements on the receivers such that the signals received at different antenna elements are uncorrelated.[1] As a result, the receive-antenna correlation matrix becomes the identity matrix, i.e., $\mathbf{\Phi}_{xy} = \mathbf{I}$. Now, the channel model can be written as

$$\mathbf{H}_{xy} = \mathbf{Z}_{xy} \mathbf{\Sigma}_{xy}^{1/2}. \tag{2}$$

[1]Our results can be extended to the case where the channel has double-sided correlation structure (i.e., to the case where the signals arriving at the receiver are also correlated) as in [8]. However, this extension is omitted in this paper.

When the relay is allowed to transmit and receive at the same time, the channel is said to be in full-duplex mode. In this case, received signals at the relay and the destination are given as

$$\mathbf{r} = \mathbf{H}_{sr}\mathbf{x}_s + \mathbf{n}_r, \qquad \mathbf{y} = \mathbf{H}_{sd}\mathbf{x}_s + \mathbf{H}_{rd}\mathbf{x}_r + \mathbf{n}_y \tag{3}$$

where $\mathbf{r}$ is N_r long received vector at the relay, $\mathbf{y}$ is N_d long received vector at the destination, $\mathbf{x}_s$ is an M_s long transmitted signal from the source, and $\mathbf{x}_r$ is an M_r long transmitted signal from the relay. The covariance matrices of the transmitted signals are $\mathbf{Q}_s = E[\mathbf{x}_s\mathbf{x}_s^\dagger]$ and $\mathbf{Q}_r = E[\mathbf{x}_r\mathbf{x}_r^\dagger]$, and there are individual power constraints on the source and relay transmit covariance matrices. Noise vectors at the relay, i.e., $\mathbf{n}_r$, and at the destination, i.e., $\mathbf{n}_y$ are zero-mean identity covariance complex Gaussian random vectors.

In half-duplex transmission, the relay cannot transmit and receive signals simultaneously. Therefore, one transmission frame is divided into two phases. Correspondingly, source input is also divided into two parts. In the first phase, the relay behaves as a receiver only, and the source transmits the first part of its input $\mathbf{x}_s^{(1)}$. In this phase, the received signals at the relay and destination are

$$\mathbf{r} = \mathbf{H}_{sr}\mathbf{x}_s^{(1)} + \mathbf{n}_r, \qquad \mathbf{y}_1 = \mathbf{H}_{sd}\mathbf{x}_s^{(1)} + \mathbf{n}_y^{(1)} \tag{4}$$

where the covariance matrix of $\mathbf{x}_s^{(1)}$ is $\mathbf{Q}_s^{(1)} = E[\mathbf{x}_s^{(1)}\mathbf{x}_s^{(1)\dagger}]$. In the second phase, the relay behaves as a transmitter. The source transmits the second part of its input $\mathbf{x}_s^{(2)}$ and the relay transmits $\mathbf{x}_r$. In this phase, the received signal at the destination is

$$\mathbf{y}_2 = \mathbf{H}_{sd}\mathbf{x}_s^{(2)} + \mathbf{H}_{rd}\mathbf{x}_r + \mathbf{n}_y^{(2)} \tag{5}$$

where the covariance matrix of $\mathbf{x}_s^{(2)}$ is $\mathbf{Q}_s^{(2)} = E[\mathbf{x}_s^{(2)}\mathbf{x}_s^{(2)\dagger}]$, and the noise vectors at the destination, i.e., $\mathbf{n}_y^{(1)}$ and $\mathbf{n}_y^{(2)}$, are zero-mean identity covariance complex Gaussian random vectors.

III. Matrix Differential Calculus

Here, we introduce the matrix differential calculus [11] that will be useful later. We start by defining the "differential" of a scalar function. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a real-valued function. The differential is the linear part of the increment of the value of a function, i.e., $\phi(x+u) - \phi(x)$, at a fixed-point x with increment u. The derivative of function ϕ at point x is found by dividing the differential of the function with increment u, and by taking the limit as u goes to 0, i.e.,

$$\phi'(x) = \lim_{u \to 0} \frac{\phi(x+u) - \phi(x)}{u}.$$

The differential is denoted by $d\phi(x;u)$, and it is equal to $d\phi(x;u) = u\phi'(x)$. Similarly, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function, and $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$. The differential of f is defined as $df(\mathbf{x};\mathbf{u}) = \mathbf{A}(\mathbf{x})\mathbf{u}$, where $m \times n$-dimensional matrix $\mathbf{A}(\mathbf{x})$ is called the first derivative of f at $\mathbf{x}$. It is important to note here that, while the differential of a vector-valued function is a vector, the derivative of a vector-valued function is a matrix.

Since dealing with a matrix is cumbersome, partial derivatives are often used in optimization problems involving vector-valued functions. In fact, as the *first identification theorem* in [11] states, the elements of $m \times n$ matrix $\mathbf{A}(\mathbf{x})$ are the partial derivatives of f evaluated at $\mathbf{x}$, and $\mathbf{A}(\mathbf{x})$ is called the Jacobian matrix of f, i.e., $Df(\mathbf{x}) = \mathbf{A}(\mathbf{x})$. As a result of this, if f is differentiable at $\mathbf{x}$ and we have found a differential df at $\mathbf{x}$, then the value of the partial derivatives at $\mathbf{x}$ can be immediately determined.

Finally, the differential of a matrix-valued function can be determined using the vector representation of matrices. Let $F : \mathbb{R}^{n\times q} \to \mathbb{R}^{m\times p}$ be a matrix function and differentiable at $\mathbf{X} \in \mathbb{R}^{n\times q}$. Then, the differential can be written as $\mathrm{vec} dF(\mathbf{X}; \mathbf{U}) = \mathbf{A}(\mathbf{X})\mathrm{vec}\mathbf{U}$, where the Jacobian is an $mp \times nq$ matrix $DF(\mathbf{X}) = \mathbf{A}(\mathbf{X})$.

Given matrix function $F(\mathbf{X})$, determining the derivative of this function from its differential is carried out as follows: 1) Compute the differential of $F(\mathbf{X})$; 2) vectorize the latter result to obtain $d\ \mathrm{vec}\ F(\mathbf{X}) = \mathbf{A}(\mathbf{X})d\ \mathrm{vec}\ \mathbf{X}$; and 3) conclude that $DF(\mathbf{X}) = \mathbf{A}(\mathbf{X})$. In this paper, we mainly deal with scalar functions $\phi : \mathbb{R}^{n\times q} \to \mathbb{R}$ of matrix variables. In this case, the derivative can be written as

$$D\phi(\mathbf{X}) = \frac{\partial \phi(\mathbf{X})}{\partial (\mathrm{vec}\ \mathbf{X}^T)}. \tag{6}$$

However, the idea of arranging the partial derivatives of $\phi(\mathbf{X})$ into a matrix (rather than a vector) is appealing and sometimes useful; therefore, with a slight abuse of notation, we will use

$$D\phi(\mathbf{X}) = \frac{\partial \phi(\mathbf{X})}{\partial \mathbf{X}}. \tag{7}$$

For scalar functions of the matrix variable, the differential of $\phi(\mathbf{X})$ is given as [11] $d\phi = (\mathrm{vec}\mathbf{A})^T d\ \mathrm{vec}\ \mathbf{X}$, which is also equal to $d\phi = \mathrm{tr}(\mathbf{A}^T d\mathbf{X})$, where the Jacobian matrix is obtained as $D\phi(\mathbf{X}) = (\partial\phi(\mathbf{X})/\partial\mathbf{X}) = \mathbf{A}$.

Using this, we now give some important differentials that will be useful later. Differential of $\mathrm{tr}(\mathbf{X})$ with respect to $\mathbf{X}$ can be calculated as $d\,\mathrm{tr}(\mathbf{X}) = \mathrm{tr}(d\mathbf{X})$. Therefore, the derivative of $\mathrm{tr}(\mathbf{X})$ is $D\mathrm{tr}(\mathbf{X}) = \mathbf{I}$. Given matrix $\mathbf{H}$, the differential with respect to $\mathbf{X}$ of the expression $\log|\mathbf{I} + \mathbf{H}\mathbf{X}\mathbf{H}^\dagger|$ can be calculated as $d\log|\mathbf{I} + \mathbf{H}\mathbf{X}\mathbf{H}^\dagger| = \mathrm{tr}(\mathbf{H}^\dagger(\mathbf{I} + \mathbf{H}\mathbf{X}\mathbf{H}^\dagger)^{-1}\mathbf{H}d\mathbf{X})$. Therefore, the derivative of the expression is $D\log|\mathbf{I} + \mathbf{H}\mathbf{X}\mathbf{H}^\dagger| = \mathbf{H}^\dagger(\mathbf{I} + \mathbf{H}\mathbf{X}\mathbf{H}^\dagger)^{-1}\mathbf{H}$.

IV. Capacity Bounds in Full-Duplex Relaying

Although the capacity is not known for general fading MIMO relay channels, capacity bounds can be derived using [2]. Similar bounds are derived for fading MIMO systems before [5], [6], in the form of optimization problems that need to be solved. Our main contribution here is solving the max–min optimization problems for full-duplex transmission to evaluate the DF achievable rate and the cut-set upper bound. Solutions to these optimization problems are not trivial when the transmitters have the covariance information on the channel, which is assumed in this paper.

A. Lower Bound on the Capacity

The DF achievable rate can be derived from the mutual information expressions in [2]. These expressions are evaluated for MIMO relay systems in [6] for the case where the transmitters do not have any information about the channel and the source and relay share the same power constraint. In that case, lower bound maximizing transmit covariance matrices are identity matrices, and the cross-correlation matrices are zero [5]. Here, we evaluate the DF achievable rate when the transmitters have the covariance information on the channel, and the source and relay have individual power constraints as it is assumed for single-antenna systems in [4]. This rate, which is given in Theorem 1 below, is in terms of a max–min-type optimization problem over the source and relay transmit covariance matrices. Later, we solve this optimization problem and propose an iterative algorithm that gives the transmit covariance matrices.

Theorem 1: When there is only channel covariance information at the transmitters and perfect CSI at the receivers, the DF achievable rate of a full-duplex MIMO relay channel is given as

$$C_{\mathrm{fd}} \geq \max_{\mathrm{tr}(\mathbf{Q}_s)\leq P_s, \mathrm{tr}(\mathbf{Q}_r)\leq P_r} \min(I_{\mathrm{mac}}, I_{sr}) \tag{8}$$

$$I_{\mathrm{mac}} = E\left[\log\left|\mathbf{I} + \mathbf{H}_{sd}\mathbf{Q}_s\mathbf{H}_{sd}^\dagger + \mathbf{H}_{rd}\mathbf{Q}_r\mathbf{H}_{rd}^\dagger\right|\right] \tag{9}$$

$$I_{sr} = E\left[\log\left|\mathbf{I} + \mathbf{H}_{sr}\mathbf{Q}_s\mathbf{H}_{sr}^\dagger\right|\right]. \tag{10}$$

Proof: Using block Markov coding, the DF achievable rate is given as [2, Sec. VI]

$$R = \max_{p(\mathbf{x}_s,\mathbf{x}_r)} \min\left(I(\mathbf{x}_s;\mathbf{r}|\mathbf{x}_r), I(\mathbf{x}_s,\mathbf{x}_r;\mathbf{y})\right) \tag{11}$$

where $I(\mathbf{x}_s,\mathbf{x}_r;\mathbf{y}) = E[I(\mathbf{x}_s,\mathbf{x}_r;\mathbf{y}|\mathbf{H}_{sd},\mathbf{H}_{rd})]$, $I(\mathbf{x}_s;\mathbf{r}|\mathbf{x}_r) = E[I(\mathbf{x}_s;\mathbf{r}|\mathbf{x}_r,\mathbf{H}_{sr})]$, $\mathbf{x}_s$ and $\mathbf{x}_r$ are circularly symmetric complex Gaussian random vectors, and $p(\mathbf{x}_s,\mathbf{x}_r)$ is the joint distribution of these random vectors. To prove the theorem, we have to calculate the mutual information expressions of $I(\mathbf{x}_s,\mathbf{x}_r;\mathbf{y})$ and $I(\mathbf{x}_s;\mathbf{r}|\mathbf{x}_r)$. The former is calculated in [6] as

$$I(\mathbf{x}_s,\mathbf{x}_r;\mathbf{y}) \leq E\left[\log\left|\mathbf{I} + [\mathbf{H}_{sd}\ \mathbf{H}_{rd}]\begin{bmatrix}\mathbf{Q}_s & \mathbf{Q}_{sr}\\ \mathbf{Q}_{rs} & \mathbf{Q}_r\end{bmatrix}[\mathbf{H}_{sd}\ \mathbf{H}_{rd}]^\dagger\right|\right] \tag{12}$$

In (12), equality is achieved when the input distributions are Gaussian, which is the case in this paper. As pointed out in [5], the cross-correlation matrices $\mathbf{Q}_{sr} = E[\mathbf{x}_s{\mathbf{x}_r}^\dagger]$ and $\mathbf{Q}_{rs} = E[\mathbf{x}_r{\mathbf{x}_s}^\dagger]$ that maximize the mutual information values are zero when the transmitters do not know more than the statistics of the channel. The insight behind this is explained in [5] as follows. In the integral operation in (12), one can replace $\mathbf{H}_{sd}$ with $-\mathbf{H}_{sd}$ since the beginning phase of the integration is not important for any $\mathbf{H}_{sd}$. This is equivalent to keeping $\mathbf{H}_{sd}$ the same but replacing $\mathbf{x}_s$ with $-\mathbf{x}_s$. However, this sign change makes cross-correlation matrices to also change their sign. Using the concavity of mutual information, if the cross-correlation matrices are chosen to be zero, the mutual information cannot decrease. Thus, the signals are independent [5]. After taking the cross-correlation matrices to be zero matrices, (12) becomes (9).

The single-user capacity from source to relay, i.e., $I(\mathbf{x}_s;\mathbf{r}|\mathbf{x}_r)$, is calculated in [6] as

$$I(\mathbf{x}_s;\mathbf{r}|\mathbf{x}_r)\leq E\left[\log\left|\mathbf{I}+\mathbf{H}_{sr}\left(\mathbf{Q}_s-\mathbf{Q}_{sr}\mathbf{Q}_r^{-1}\mathbf{Q}_{rs}^{\dagger}\right)\mathbf{H}_{sr}^{\dagger}\right|\right]. \quad (13)$$

The equality in (13) is achieved when the input distributions are Gaussian, which is the case in this paper. Using the fact that $\mathbf{Q}_{sr}=\mathbf{0}$ and $\mathbf{Q}_{rs}=\mathbf{0}$ [5] again, (13) becomes (10). □

Theorem 1 gives the DF achievable rate in terms of a max–min-type optimization problem that still needs to be solved. The solution to this problem requires a joint optimization over the source and relay transmit covariance matrices, because the optimum $\mathbf{Q}_s$ that maximizes I_{mac} in (9) and the optimum $\mathbf{Q}_s$ that maximizes I_{sr} in (10) are different. If we maximize I_{mac}, that choice of $\mathbf{Q}_s$ will result in low I_{sr}. As a result, I_{sr} will come out of the minimization in (8), and the achievable rate will attain a lower value. In the same way, if we maximize I_{sr}, that choice of $\mathbf{Q}_s$ will result in a low I_{mac}. To solve this tradeoff, $\mathbf{Q}_s$ and $\mathbf{Q}_r$ should be found jointly.

We utilize a method that is proposed in [4]. In this method, the following function $\mathbf{R}_{\text{fl}}$ of α and $\mathbf{Q}$ is defined as

$$\mathbf{R}_{\text{fl}}(\alpha,\mathbf{Q})=\alpha I_{\text{mac}}(\mathbf{Q})+(1-\alpha)I_{sr}(\mathbf{Q}), \quad 0\leq\alpha\leq 1 \quad (14)$$

where $\mathbf{Q}=[\mathbf{Q}_s \quad \mathbf{Q}_r]$. The max–min problem in (8) corresponds to first maximizing $\mathbf{R}_{\text{fl}}(\alpha,\mathbf{Q})$ over $\mathbf{Q}$ for a fixed α, and then taking the minimum over α [4]. It is important to note that [4] applied this method for a different channel assumption, in particular when both the transmitters and the receivers know the CSI. Under this assumption, [4] solved the max–min problem. In this paper, we apply the same method, but since our CSI assumption is different, the solution of the max–min problem is completely different and more complex, and it results in an iterative algorithm.

Let us define $\mathbf{V}_{\text{fl}}(\alpha)$ as $\mathbf{V}_{\text{fl}}(\alpha)=\max_{\mathbf{Q}}\mathbf{R}(\alpha,\mathbf{Q})$ and suppose that α^* provides the minimum value of $\mathbf{V}_{\text{fl}}(\alpha)$. Depending on the value of α^*, we have three cases. Optimum source and relay covariance matrices may be different in all three cases. In the first case $(\alpha^*=0)$, $\mathbf{R}_{\text{fl}}(0,\mathbf{Q})=I_{sr}(\mathbf{Q})$ and condition $I_{\text{mac}}(\mathbf{Q})\geq I_{sr}(\mathbf{Q})$ should be satisfied [4]. Since the achievable rate is found by maximizing $I_{sr}(\mathbf{Q})$ only, we find the source transmit covariance matrix $\mathbf{Q}_s$ as a solution to the point-to-point problem from the source to the relay. When the receiver knows perfect CSI and the transmitter knows partial CSI, a point-to-point problem is already solved in [9]. Then, we find the relay transmit covariance matrix $\mathbf{Q}_r$ by maximizing $I_{\text{mac}}(\mathbf{Q})$ with a fixed $\mathbf{Q}_s$. This is also equivalent to a single-user problem, which is solved in [9] and, therefore, omitted here.

In the second case $(\alpha^*=1)$, $\mathbf{R}_{\text{fl}}(1,\mathbf{Q})=I_{\text{mac}}(\mathbf{Q})$ and condition $I_{\text{mac}}(\mathbf{Q})\leq I_{sr}(\mathbf{Q})$ should be satisfied. In this case, the achievable rate is found by maximizing $I_{\text{mac}}(\mathbf{Q})$, which is a MAC problem. When the receiver knows perfect CSI and the transmitters know partial CSI, the MIMO-MAC system is already solved in [9].

In the third case $(0<\alpha^*<1)$, $\mathbf{R}_{\text{fl}}(\alpha^*,\mathbf{Q})=\alpha^* I_{\text{mac}}(\mathbf{Q})+(1-\alpha^*)I_{sr}(\mathbf{Q})$ and condition $I_{\text{mac}}(\mathbf{Q})=I_{sr}(\mathbf{Q})$ should be satisfied. In this case, we find the transmit covariance matrices of the source and relay as functions of α^*. The third case is the most interesting case as the solution is not trivial. In that case, $\mathbf{Q}_s$ and $\mathbf{Q}_r$ must be optimized jointly since objective function $R_{\text{fl}}(\alpha,\mathbf{Q})$ includes both I_{sr} and I_{mac}. However, this joint optimization problem cannot be solved by using the methods from the previous literature. In studies such as [7]–[9], the transmit covariance matrices are always found by determining their eigenvectors first. This reduces the problem of finding the eigenvalues of the transmit covariance matrix from a matrix variable to a vector (and sometimes scalar) variable problem. Since the eigenvectors cannot be determined in closed form in this joint optimization, one needs to come up with another solution.

It is always possible to solve this joint optimization problem using classical convex optimization methods [13]. The disadvantage of classical convex optimization methods is that they are extremely slow and, therefore, cannot be used in a slow fading wireless environment, where the statistics of the channel changes slowly. However, under certain assumptions on the channel, it might be possible to choose eigenvectors of the transmit covariance matrices cleverly and propose fast and efficient algorithms to find the eigenvalues. One such assumption is that the source-to-destination link is weaker than the source-to-relay link. Therefore, the source node chooses to transmit along the eigenvectors of the covariance of the source-to-relay channel, instead of the jointly optimal directions. Jointly optimal directions are possibly a combination of the eigenvectors of the covariance of the source-to-relay channel and those of the source-to-destination channel. In vague terms, the source node chooses to transmit toward the relay.

Once the transmit directions of the source node is given, the transmit directions of the relay node can be found as the eigenvectors of the relay-to-destination channel [10]. Having chosen the eigenvectors (i.e., transmit directions) of the source and relay transmit covariance matrices, then one can find the jointly optimum power values allocated along these transmit directions by modifying the methods previously offered in the literature. Clearly, this solution is suboptimal. Although we omit the details of this derivation [10], we will compare the performance of this solution to the optimum solution in Section VI.

The optimal solution uses matrix differential calculus. First, (14) will be maximized over $\mathbf{Q}$ for a fixed α^*, $0<\alpha^*<1$. Note that, transmit covariance matrices that will result from this optimization will depend on α^* as follows:

$$\mathbf{V}_{\text{fl}}(\alpha^*)=\max_{\text{tr}(\mathbf{Q}_s)\leq P_s,\text{tr}(\mathbf{Q}_r)\leq P_r}\left(\alpha^* I_{\text{mac}}(\mathbf{Q})+(1-\alpha^*)I_{sr}(\mathbf{Q})\right). \quad (15)$$

The Lagrangian of (15) can be written as

$$L=\mathbf{R}_{\text{fl}}(\alpha^*,\mathbf{Q})-\mu_s\left(\text{tr}(\mathbf{Q}_s)-P_s\right)-\mu_r\left(\text{tr}(\mathbf{Q}_r)-P_r\right) \quad (16)$$

where μ_s and μ_r are Lagrange multipliers corresponding to source and relay power constraints, respectively. Here, we will directly take the derivative of (16) with respect to $\mathbf{Q}_s$ and $\mathbf{Q}_r$. By using matrix differential calculus and referring to the examples in Section III, one can take the derivative of (16) with respect to $\mathbf{Q}_s$ and $\mathbf{Q}_r$ to obtain the following Karush–Kuhn–Tucker (KKT) conditions:

$$E\left[\alpha^*\mathbf{H}_{sd}^{\dagger}\mathbf{D}_{\text{mac}}^{-1}\mathbf{H}_{sd}+(1-\alpha^*)\mathbf{H}_{sr}^{\dagger}\mathbf{D}_{sr}^{-1}\mathbf{H}_{sr}\right]\leq\mu_s\mathbf{I} \quad (17)$$

$$E\left[\alpha^*\mathbf{H}_{rd}^{\dagger}\mathbf{D}_{\text{mac}}^{-1}\mathbf{H}_{rd}\right]\leq\mu_r\mathbf{I} \quad (18)$$

where $\mathbf{D}_{\mathrm{mac}}$ is the expression inside the determinant in (9), and $\mathbf{D}_{sr}$ is the expression inside the determinant in (10). Note that we omitted the complementary slackness conditions while writing KKT conditions. The KKT conditions in (17) and (18) are satisfied with equality when matrices $\mathbf{Q}_s$ and $\mathbf{Q}_r$ are positive definite, respectively. Otherwise, KKT conditions are satisfied with strict inequalities. To solve for $\mathbf{Q}_r$ and $\mathbf{Q}_s$, we need equalities. Therefore, we utilize the reasoning that is first introduced in [9]. Let us denote the left-hand side of (17) as E_1 and the left-hand side of (18) as E_2. We multiply both sides of (17) with $\mathbf{Q}_s$ from the right-hand side and both sides of (18) with $\mathbf{Q}_r$ from the right-hand side; thus, we have

$$E_1\mathbf{Q}_s = \mu_s\mathbf{Q}_s \tag{19}$$

$$E_2\mathbf{Q}_r = \mu_r\mathbf{Q}_r. \tag{20}$$

We note that when all inequalities in (17) and (18) are equalities, then (19) and (20) follows directly. When all inequalities in (17) and (18) are strict inequalities, then $\mathbf{Q}_s = \mathbf{0}$ and $\mathbf{Q}_r = \mathbf{0}$. Therefore, both sides of (19) and (20) are zero. When some inequalities in (17) and (18) are equalities and some are strict inequalities, then $\mathbf{Q}_s$ and $\mathbf{Q}_r$ have blocks of zero matrices corresponding to the locations of strict inequalities. In that case, it is possible to separate the transmit covariance matrices into two parts: a positive definite part and a zero part. The former already satisfies (17) and (18) with equalities, and the latter satisfies (19) and (20) with equalities. As a result, unlike (17) and (18), (19) and (20) are always satisfied with equality for optimum transmit covariance matrices. By applying the trace operator, Lagrange multipliers are calculated as

$$\mu_s = \frac{\mathrm{tr}(E_1\mathbf{Q}_s)}{P_s}, \qquad \mu_r = \frac{\mathrm{tr}(E_2\mathbf{Q}_r)}{P_r}. \tag{21}$$

By substituting these μ_s and μ_r into (19) and (20), we find the fixed-point equations, which have to be satisfied by the optimum transmit covariance matrices as follows:

$$\mathbf{Q}_s = \frac{E_1\mathbf{Q}_s}{\mathrm{tr}(E_1\mathbf{Q}_s)}P_s, \qquad \mathbf{Q}_r = \frac{E_2\mathbf{Q}_r}{\mathrm{tr}(E_2\mathbf{Q}_r)}P_r. \tag{22}$$

We propose the following iterative algorithm to solve for the fixed-point equations that are obtained from (19) and (20):

$$\mathbf{Q}_s(n+1) = \frac{E_1(n)\mathbf{Q}_s(n)P_s}{\mathrm{tr}\left(E_1(n)\mathbf{Q}_s(n)\right)}, \mathbf{Q}_r(n+1) = \frac{E_2(n)\mathbf{Q}_r(n)P_r}{\mathrm{tr}\left(E_2(n)\mathbf{Q}_r(n)\right)} \tag{23}$$

This iterative algorithm finds the optimum transmit covariance matrices of the source and relay for Case 3. After running this algorithm for different α values, a minimization over α is performed to find the lower bound. It is important to note that the algorithm in (23) updates every element of the transmit covariance matrices at once. As aforementioned, the eigenvectors of the transmit covariance matrices were not determined beforehand; they are found implicitly after the algorithm in (23) converges.

The convergence of the algorithm in (23) is an important issue. Due to mathematical complexity, convergence analysis of the algorithm seems intractable. However, we observe through numerous simulations that the algorithm converges, regardless of the initial points.

B. Upper Bound on the Capacity

Having derived the DF achievable rate and jointly optimized the source and transmitter covariance matrices, here, we consider the cut-set upper bound. This bound is introduced in [2] and evaluated for different channel model assumptions in the literature. For example, when the receivers have perfect CSI and the transmitters have no CSI, the cut-set upper bound on the MIMO relay channel capacity is found in [6]. In this paper, we consider a case where there is transmit covariance information at the transmitters. In this case, similar to the lower bound development, we first evaluate the mutual information expressions in the cut-set bound and then optimize the upper bound over $\mathbf{Q}_s$ and $\mathbf{Q}_r$.

Theorem 2: When there is only channel covariance information at the transmitters and perfect CSI at the receivers, the cut-set upper bound of a full-duplex MIMO relay channel is given as

$$C_{\mathrm{fd}} \leq \max_{\mathrm{tr}(\mathbf{Q}_s)\leq P_s, \mathrm{tr}(\mathbf{Q}_r)\leq P_r} \min(I_{\mathrm{mac}}, I_{\mathrm{bc}}) \tag{24}$$

where $I_{\mathrm{bc}} = E[\log|\mathbf{I} + \mathbf{H}_{\mathrm{bc}}\mathbf{Q}_s\mathbf{H}_{\mathrm{bc}}^{\dagger}|]$, I_{mac} is given in (9), and $\mathbf{H}_{\mathrm{bc}} = [\mathbf{H}_{sd}^{\dagger} \quad \mathbf{H}_{sr}^{\dagger}]^{\dagger}$.

The proof of Theorem 2 is very similar to the proof of Theorem 1, and it is omitted here due to space restrictions and due to the fact that the contribution of this paper is not the evaluation of the upper bound expression but is providing its solution. The proof basically calculates the cut-set upper bound with zero cross-correlation matrices. Note that the DF achievable rate and the cut-set upper bound expressions both involve the same I_{mac}. Therefore, the lower and upper bounds meet and provide the capacity if I_{mac} comes out of the minimization in both cases.

As in the case of the lower bound, we also have a max–min problem to solve in the upper bound. The method for this solution is similar to the lower bound solution and utilizes matrix differential calculus. We skip some of the development where it can easily be obtained from lower bound analysis. This time, we define $\mathbf{R}_{\mathrm{fu}}$ as

$$\mathbf{R}_{\mathrm{fu}}(\alpha, \mathbf{Q}) = \alpha I_{\mathrm{mac}}(\mathbf{Q}) + (1-\alpha)I_{\mathrm{bc}}(\mathbf{Q}), \quad 0 \leq \alpha \leq 1. \tag{25}$$

Note that unlike the DF achievable rate, the upper bound $\mathbf{R}_{\mathrm{fu}}$ depends on I_{bc} and not on I_{sr}. Depending on the value of minimum α^*, the solution again has three cases. In the first case, ($\alpha^* = 0$), $\mathbf{R}(0, \mathbf{Q}) = I_{\mathrm{bc}}(\mathbf{Q})$, and condition $I_{\mathrm{mac}}(\mathbf{Q}) \geq I_{\mathrm{bc}}(\mathbf{Q})$ should be satisfied. For this case, the Lagrangian can be written as

$$L = I_{\mathrm{bc}}(\mathbf{Q}) - \mu_s\left(\mathrm{tr}(\mathbf{Q}_s) - P_s\right) \tag{26}$$

Using matrix differential calculus and by taking the derivative of (26) with respect to $\mathbf{Q}_s$, we obtain the KKT conditions. Then, similar to the lower bound, we derive the following algorithm:

$$\mathbf{Q}_s(n+1) = \frac{E_3(n)\mathbf{Q}_s(n)}{\mathrm{tr}\left(E_3(n)\mathbf{Q}_s(n)\right)}P_s \tag{27}$$

where $E_3 = E[\mathbf{H}_{\mathrm{bc}}^{\dagger}\mathbf{D}_{\mathrm{bc}}^{-1}\mathbf{H}_{\mathrm{bc}}]$, and $\mathbf{D}_{\mathrm{bc}}$ is the matrix inside the determinant of I_{bc}. Next, $\mathbf{Q}_r$ is found by maximizing I_{mac}

using fixed $\mathbf{Q}_s$ found earlier. This is equivalent to a single-user problem that is solved in [9].

The second case is again a MIMO-MAC channel and is already known. In the third case ($0 < \alpha^* < 1$), $\mathbf{R}_{\rm fu}(\alpha^*, \mathbf{Q}) = \alpha^* I_{\rm mac}(\mathbf{Q}) + (1-\alpha^*) I_{\rm bc}(\mathbf{Q})$, and $I_{\rm mac}(\mathbf{Q}) = I_{\rm bc}(\mathbf{Q})$ should be satisfied. The Lagrangian for this case is given as

$$L = \mathbf{R}_{\rm fu}(\alpha^*, \mathbf{Q}) - \mu_s\left(\mathrm{tr}(\mathbf{Q}_s) - P_s\right) - \mu_r\left(\mathrm{tr}(\mathbf{Q}_r) - P_r\right). \quad (28)$$

Using matrix differential calculus and by taking the derivative of (28) with respect to $\mathbf{Q}_s$ and $\mathbf{Q}_r$, we obtain the KKT conditions. Then, using the similar method as in the lower bound we derive the algorithm as follows:

$$\mathbf{Q}_s(n+1) = \frac{E_4(n)\mathbf{Q}_s(n)P_s}{\mathrm{tr}\left(E_4(n)\mathbf{Q}_s(n)\right)}, \; \mathbf{Q}_r(n+1) = \frac{E_2(n)\mathbf{Q}_r(n)P_r}{\mathrm{tr}\left(E_2(n)\mathbf{Q}_r(n)\right)} \quad (29)$$

where $E_4 = E[\alpha^* \mathbf{H}_{sd}^\dagger \mathbf{D}_{\rm mac}^{-1} \mathbf{H}_{sd} + (1-\alpha^*)\mathbf{H}_{\rm bc}^\dagger \mathbf{D}_{\rm bc}^{-1} \mathbf{H}_{\rm bc}]$. This iterative algorithm finds the transmit covariance matrices of the source and relay nodes that solves the Case 3 of the optimization problem in the upper bound. Finally, a minimization over α is performed to find which case results in the upper bound.

V. Capacity Bounds in Half-Duplex Relaying

In Section IV, we considered full-duplex transmission where the relay was assumed to receive and transmit at the same time. However, it might be difficult to implement full-duplex transmission in practice. Here, we consider a half-duplex transmission where the transmission block is divided into two phases. In the first phase, the relay receives the signal, and in the second phase, it transmits. The system model was given in (4) and (5). The DF achievable rate and the cut-set upper bound are derived for half-duplex channels in [3] and [4] for single-antenna systems and in [14] for MIMO systems. Here, we generalize these bounds to the case where the transmitters have the covariance information on the channel. Then, we find the source and transmit covariance matrices that achieve those bounds.

A. Lower Bound on the Capacity

In DF half-duplex transmission, the relay listens to the source in the first phase, decodes the message, and cooperates with the source in the second phase. Let us assume that the first phase has duration t, and the second phase has duration $1-t$; then, we have the following theorem.

Theorem 3: When there is only channel covariance information at the transmitters and perfect CSI at the receivers, the DF achievable rate of a half-duplex MIMO relay channel is

$$C_{\rm hd} \geq \max_{\substack{t\mathrm{tr}\left(\mathbf{Q}_s^{(1)}\right) + (1-t)\mathrm{tr}\left(\mathbf{Q}_s^{(2)}\right) \leq P_s \\ (1-t)\mathrm{tr}(\mathbf{Q}_r) \leq P_r \\ 1 \geq t \geq 0}} \min(I_A, I_B) \quad (30)$$

where $I_A = tE[\log|\mathbf{I} + \mathbf{H}_{sr}\mathbf{Q}_s^{(1)}\mathbf{H}_{sr}^\dagger|] + (1-t)E[\log|\mathbf{I} + \mathbf{H}_{sd}\mathbf{Q}_s^{(2)}\mathbf{H}_{sd}^\dagger|]$, and $I_B = tE[\log|\mathbf{I} + \mathbf{H}_{sd}\mathbf{Q}_s^{(1)}\mathbf{H}_{sd}^\dagger|] + (1-t)E[\log|\mathbf{I} + \mathbf{H}_{sd}\mathbf{Q}_s^{(2)}\mathbf{H}_{sd}^\dagger + \mathbf{H}_{rd}\mathbf{Q}_r\mathbf{H}_{rd}^\dagger|]$.

Proof: A general lower bound for half-duplex relay channels is given in [3] and [15] as $C_{\rm hd} \geq \min(I_A, I_B)$, where $I_A = tE[I(\mathbf{x}_s^{(1)}; \mathbf{r}|\mathbf{x}_r = 0)] + (1-t)E[I(\mathbf{x}_s^{(2)}; \mathbf{y}^{(2)}|\mathbf{x}_r)]$, and $I_B = tE[I(\mathbf{x}_s^{(1)}; \mathbf{y}|\mathbf{x}_r = 0)] + (1-t)E[I(\mathbf{x}_s^{(2)}, \mathbf{x}_r; \mathbf{y}^{(2)})]$. Here, we will calculate these mutual information expressions for the system model in this paper. The first expression in I_A is the single-user capacity from the source to the relay, whereas the second expression in I_A is the single-user capacity from the source to the destination. The first expression in I_B is also the single-user capacity from the source to the destination, whereas the second expression in I_B is the MAC capacity from the source and the relay to the destination. Since all these expressions are known, we can calculate them to get I_A and I_B. Finally, the best lower bound is found by maximizing $\min(I_A, I_B)$ over power constraints and the time duration of the relay receive period. □

Theorem 3 defines the half-duplex DF achievable rate in terms of a max–min optimization problem. When the source-to-relay channel is better than the source-to-destination channel, the half-duplex achievable rate is clearly less than the full-duplex achievable rate, as $I_A < I_{sr}$ and $I_B < I_{\rm mac}$. Next, we will solve the optimization problem in (30) with the assumption that the relay transmit duration t is fixed. We analyze the effect of relay transmit duration in Section VI.

We use the same approach as in the full-duplex case. The following function $\mathbf{R}_{\rm hl}$ of α and $\mathbf{Q}$ is defined as

$$\mathbf{R}_{hl}(\alpha, \mathbf{Q}) = \alpha I_A(\mathbf{Q}) + (1-\alpha)I_B(\mathbf{Q}), \quad 0 \leq \alpha \leq 1. \quad (31)$$

Depending on the value of α^*, we have three cases. In the first case, When $\alpha^* = 1$, $\mathbf{R}_{hl}(1, \mathbf{Q}) = I_A(\mathbf{Q})$, and $I_A(\mathbf{Q}) \leq I_B(\mathbf{Q})$ has to be satisfied [4]. In that case, Lagrangian can be written as

$$L = I_A(\mathbf{Q}) - \mu_s\left(t\mathrm{tr}\left(\mathbf{Q}_s^{(1)}\right) + (1-t)\mathrm{tr}\left(\mathbf{Q}_s^{(2)}\right) - P_s\right). \quad (32)$$

Using matrix differential calculus and by taking the derivative of (32) with respect to $\mathbf{Q}_s^{(1)}$ and $\mathbf{Q}_s^{(2)}$, we obtain the following KKT conditions:

$$E_5 = E\left[\mathbf{H}_{sr}^\dagger \mathbf{D}_k^{-1}\mathbf{H}_{sr}\right] \leq \mu_s \mathbf{I}, \; E_6 = E\left[\mathbf{H}_{sd}^\dagger \mathbf{D}_l^{-1}\mathbf{H}_{sd}\right] \leq \mu_s \mathbf{I} \quad (33)$$

where $\mathbf{D}_k$ is the inside of the determinant of the first expression in I_A, and $\mathbf{D}_l$ is the inside of the determinant of the second expression in I_A. Then, using the same arguments as in the full-duplex mode, we derive the following algorithm:

$$\mathbf{Q}_s^{(1)}(n+1) = \frac{E_5(n)\mathbf{Q}_s^{(1)}(n)P_s}{t\mathrm{tr}\left(E_5(n)\mathbf{Q}_s^{(1)}(n)\right) + (1-t)\mathrm{tr}\left(E_6(n)\mathbf{Q}_s^{(2)}(n)\right)} \quad (34)$$

$$\mathbf{Q}_s^{(2)}(n+1) = \frac{E_6(n)\mathbf{Q}_s^{(2)}(n)P_s}{t\mathrm{tr}\left(E_5(n)\mathbf{Q}_s^{(1)}(n)\right) + (1-t)\mathrm{tr}\left(E_6(n)\mathbf{Q}_s^{(2)}(n)\right)}. \quad (35)$$

After finding the source transmit covariance matrices, $\mathbf{Q}_r$ is calculated by maximizing I_B with source transmit covariance matrices fixed. This is equivalent to a single-user problem [9].

In the second case $\alpha^* = 0$, $\mathbf{R}_{hl}(0, \mathbf{Q}) = I_B(\mathbf{Q})$ and $I_A(\mathbf{Q}) \geq I_B(\mathbf{Q})$ has to be satisfied [4]. In this case, Lagrangian can be written as

$$L = I_B(\mathbf{Q}) - \mu_s \mathcal{R}_s - \mu_r \mathcal{R}_r \tag{36}$$

where $\mathcal{R}_s = (t\mathrm{tr}(\mathbf{Q}_s^{(1)}) + (1-t)\mathrm{tr}(\mathbf{Q}_s^{(2)}) - P_s)$, and $\mathcal{R}_r = ((1-t)\mathrm{tr}(\mathbf{Q}_r) - P_r)$. Using matrix differential calculus and by taking the derivative of (36) with respect to $\mathbf{Q}_s^{(1)}$, $\mathbf{Q}_s^{(2)}$, and $\mathbf{Q}_r$, we obtain the KKT conditions as follows:

$$E_7 = E\left[\mathbf{H}_{sd}^\dagger \mathbf{D}_m^{-1} \mathbf{H}_{sd}\right] \leq \mu_s \mathbf{I} \tag{37}$$
$$E_8 = E\left[\mathbf{H}_{sd}^\dagger \mathbf{D}_n^{-1} \mathbf{H}_{sd}\right] \leq \mu_s \mathbf{I} \tag{38}$$
$$E_9 = E\left[\mathbf{H}_{rd}^\dagger \mathbf{D}_n^{-1} \mathbf{H}_{rd}\right] \leq \mu_r \mathbf{I} \tag{39}$$

where $\mathbf{D}_m$ is the inside of the determinant of the first expression in I_B, and $\mathbf{D}_n$ is the inside of the determinant of the second expression in I_B. Then, using the same arguments as before, we can obtain the algorithm that finds $\mathbf{Q}_s^{(1)}(n+1)$, $\mathbf{Q}_s^{(2)}(n+1)$, and $\mathbf{Q}_r(n+1)$.

In the third case $0 < \alpha^* < 1$, and $R_{hl}(\alpha, \mathbf{Q})$ is maximized with the condition that $I_A(\mathbf{Q}) = I_B(\mathbf{Q})$ [4]. The Lagrangian can be written as

$$L = \mathbf{R}_{hl}(\alpha^*, \mathbf{Q}) - \mu_s \mathcal{R}_s - \mu_r \mathcal{R}_r. \tag{40}$$

Using matrix differential calculus and by taking the derivative of (40) with respect to $\mathbf{Q}_s^{(1)}$, $\mathbf{Q}_s^{(2)}$, and $\mathbf{Q}_r$, we obtain the KKT conditions as

$$\alpha^* E_5 + (1-\alpha^*) E_7 \leq \mu_s \mathbf{I} \tag{41}$$
$$\alpha^* E_6 + (1-\alpha^*) E_8 \leq \mu_s \mathbf{I} \tag{42}$$
$$(1-\alpha^*) E_9 \leq \mu_r \mathbf{I}. \tag{43}$$

Using the same arguments as before, we can obtain the algorithm that finds $\mathbf{Q}_s^{(1)}(n+1)$, $\mathbf{Q}_s^{(2)}(n+1)$, and $\mathbf{Q}_r(n+1)$.

Finally, after running these algorithms, we have to take a minimum over α and find the α^* that results in the minimum rate. As it can be seen, half-duplex algorithms are more complex than full-duplex algorithms since they involve three transmit covariance matrices. None of the cases over α can be solved using previous point-to-point or MAC results.

B. Upper Bound on the Capacity

The final contribution of this paper is deriving the cut-set upper bound for the half-duplex fading MIMO relay channel when the transmitters have partial CSI and evaluating the transmit covariance matrices that achieve the upper bound.

Theorem 4: When there is only channel covariance information at the transmitters and perfect CSI at the receivers, the cut-set upper bound of a half-duplex MIMO relay channel is given as

$$C_{\mathrm{hd}} \leq \max_{\substack{t\mathrm{tr}\left(\mathbf{Q}_s^{(1)}\right) + (1-t)\mathrm{tr}\left(\mathbf{Q}_s^{(2)}\right) \leq P_s \\ (1-t)\mathrm{tr}(\mathbf{Q}_r) \leq P_r \\ 1 \geq t \geq 0}} \min(I_C, I_B) \tag{44}$$

where $I_C = tE[\log|\mathbf{I} + \mathbf{H}_{\mathrm{bc}} \mathbf{Q}_s^{(1)} \mathbf{H}_{\mathrm{bc}}^\dagger|] + (1-t)E[\log|\mathbf{I} + \mathbf{H}_{sd} \mathbf{Q}_s^{(2)} \mathbf{H}_{sd}^\dagger|]$. The proof of Theorem 4 is similar to the proof of Theorem 3 and is omitted here due to space restrictions. The solution to the optimization problem in (44) is also similar to the solution to the problem in (30) and is also omitted here.

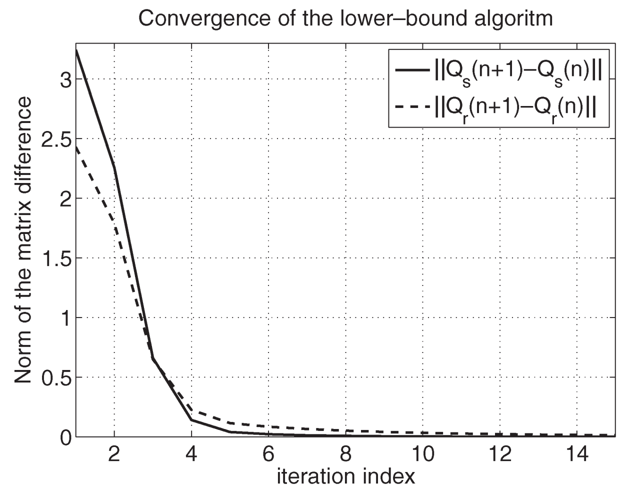

Fig. 2. Convergence of the lower bound algorithm for the half-duplex case.

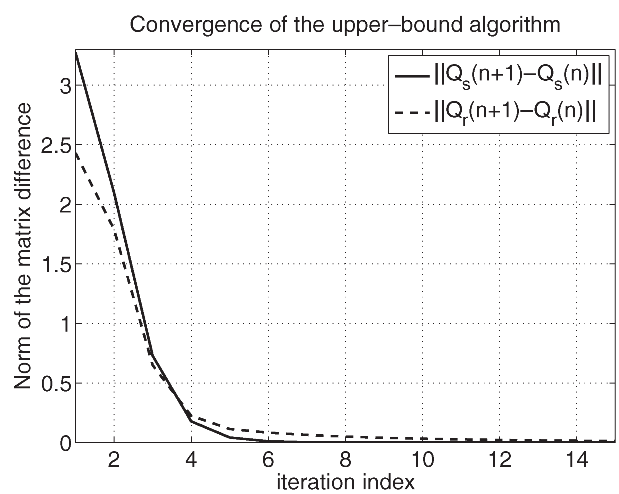

Fig. 3. Convergence of the upper bound algorithm for the half-duplex case.

VI. Numerical Results

Here, we analyze the performance of the proposed algorithms numerically. The expectation operator is calculated using Monte Carlo-type simulations. We start with a convergence analysis. This analysis is carried out for more complicated half-duplex case, and similar results can also be obtained for full-duplex case. For all calculations, the power constraints (P_s and P_r) are fixed at 10 dB, and there are three antennas at each node of the network. At each iteration of the lower bound and upper bound algorithms, we calculate the matrix norms of transmit covariance matrices of the source and relay terminals. Then, in Figs. 2 and 3, we plot the norm of the difference between two matrices of successive iterations. We clearly see that, as the iteration index increases, covariance matrices converge to their optimum values. In addition, through our

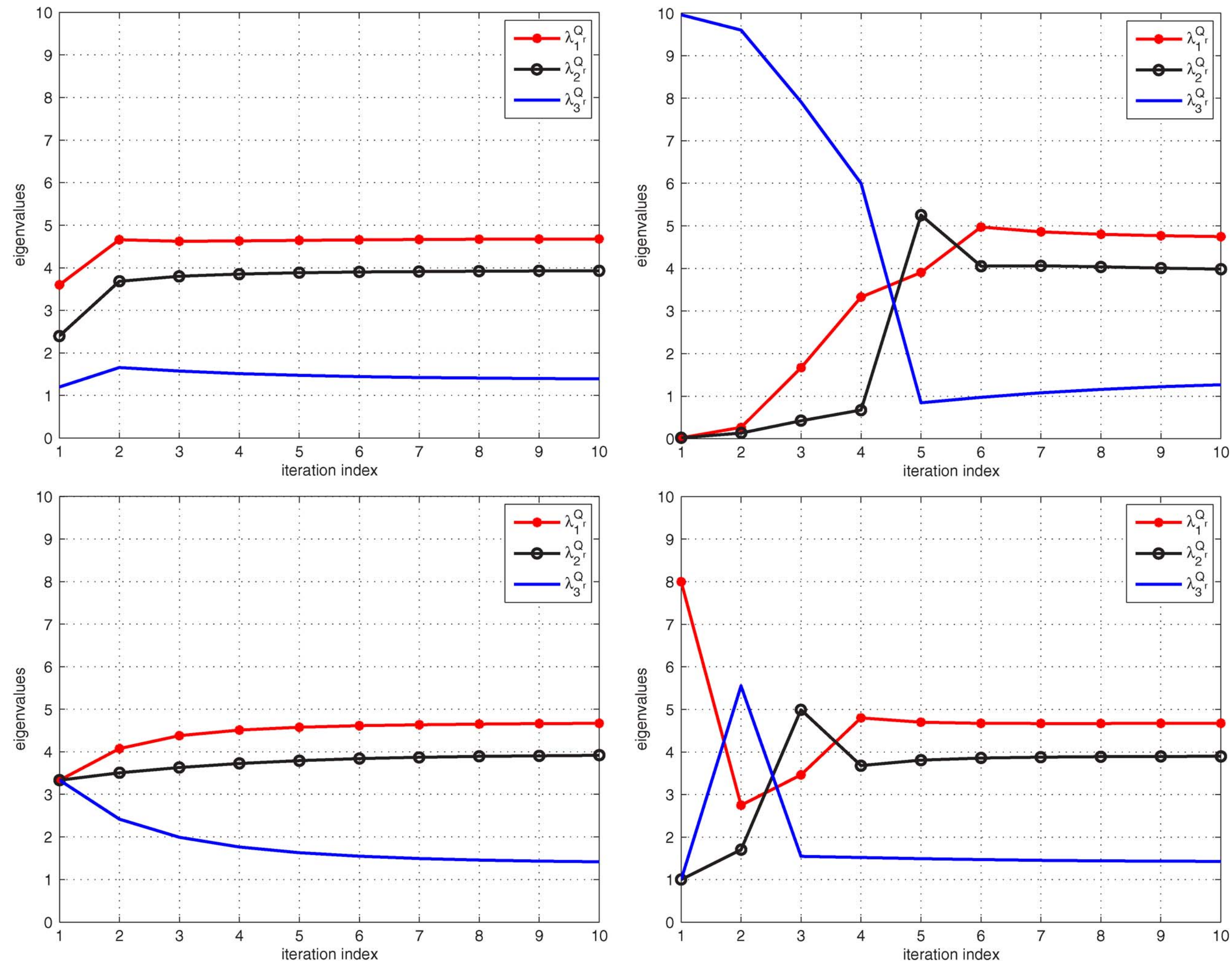

Fig. 4. Convergence rate of the eigenvalues of the relay transmit covariance matrix, starting from different initial points.

numerous trials, we have observed that the proposed algorithm converges to the same solution, regardless of the initial points. To show this, in Fig. 4, we plotted the convergence behavior starting from different initial points. Note that the points in our iteration are matrices. To keep the figures simple, we use only the eigenvalues of the relay node instead of each element of the transmit covariance matrices of both source and relay nodes. We observe that the algorithm converges faster when the eigenvalues are chosen to be as close as the eigenvalues of the channel covariance matrix (see the upper left corner in Fig. 4). When only one of the eigenvalues is chosen to be positive and the rest are chosen to be zero, the convergence is much slower. Although not shown here, similar arguments also hold for the eigenvectors. If the initial eigenvectors of the transmit covariance are chosen to be the same as the eigenvectors of the channel covariance matrix, the algorithm converges faster.

Second, capacity bounds on the full-duplex MIMO relay channel are simulated using the proposed algorithms. Power constraints are chosen to be 10 dB for all cases, and three antennas are used at all nodes. Figs. 5 and 6 calculates those bounds in bits/s/Hz for different channel covariance matrices. For the covariance matrix corresponding to Fig. 5, lower and

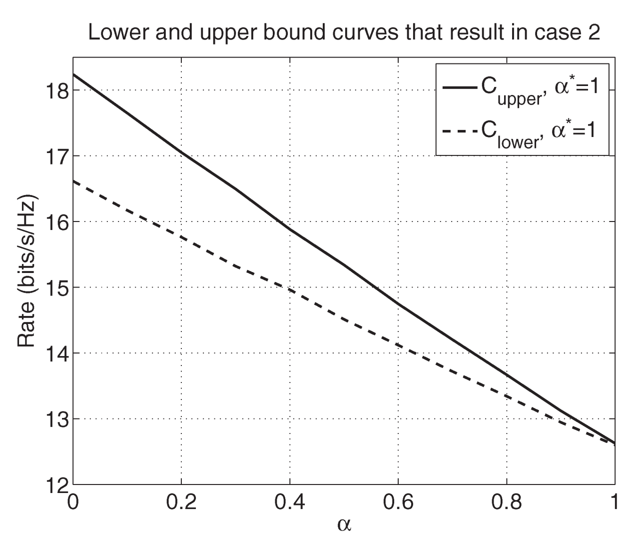

Fig. 5. Full-duplex transmission capacity lower and upper bounds that result in $\alpha^* = 1$, at which point, both curves meet and give the capacity.

upper bounds are given by $\alpha^* = 1$ point (Case 2), which is the minimum value of the curves with respect to α. As expected, the lower bound is equal to the upper bound at Case 2, and the

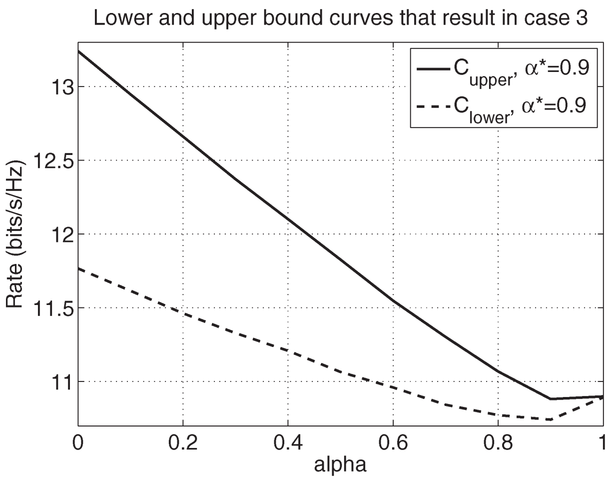

Fig. 6. Full-duplex transmission capacity lower and upper bounds that result in $\alpha^* = 0.9$.

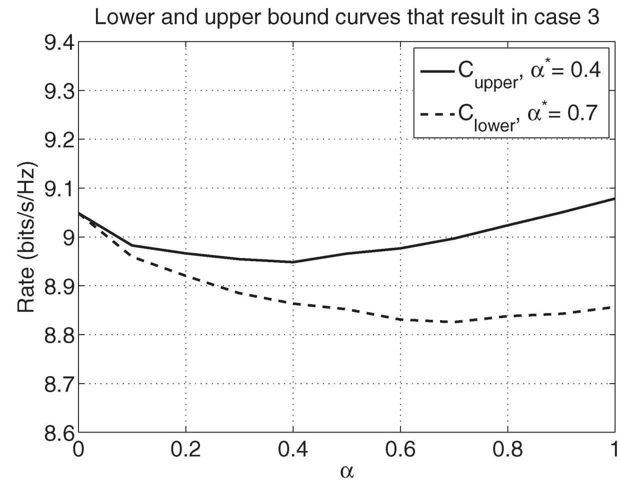

Fig. 8. Half-duplex transmission capacity lower bound that results in $\alpha^* = 0.7$ and upper bound that results in $\alpha^* = 0.4$.

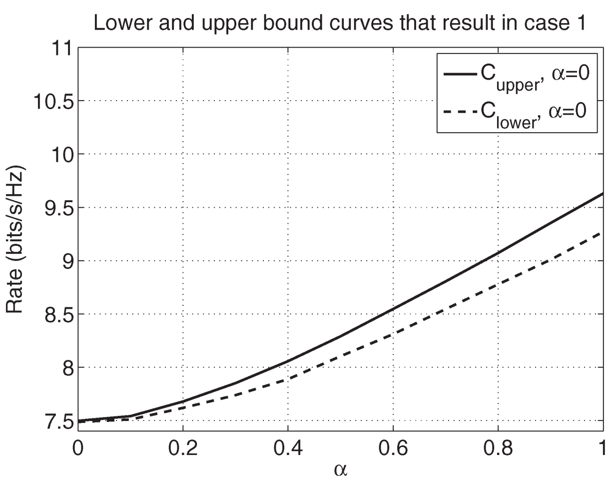

Fig. 7. Half-duplex transmission capacity lower and upper bounds that result in $\alpha^* = 0$.

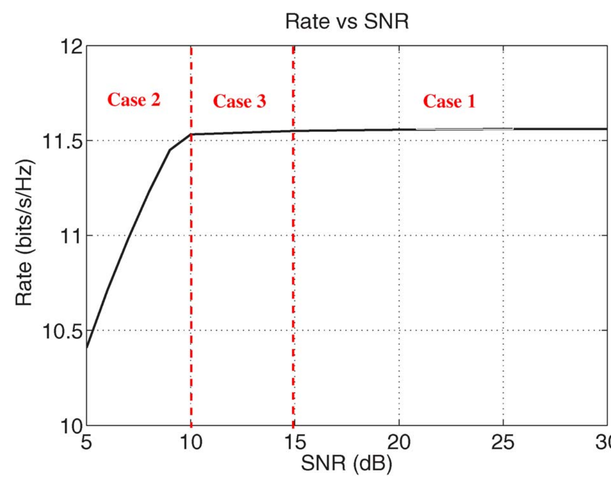

Fig. 9. Full-duplex DF achievable rate with increasing relay power.

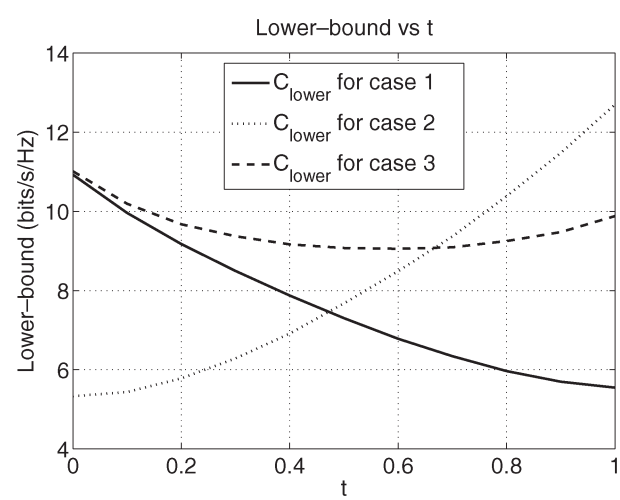

Fig. 10. Half-duplex DF achievable rate with respect to the fraction of time devoted to relay silent period.

capacity is in fact achieved for this covariance matrix setting. Similarly, for the covariance matrix corresponding to Fig. 6, lower and upper bounds are given by $\alpha^* = 0.9$ point (Case 3), which is the minimum value of the curves with respect to α. The difference between the lower and upper bounds for this case is about 1%. The maximum difference between the bounds happens in Case 1, the point of $\alpha = 0$. At that point, the difference between the rates is 10%. In addition, we observe that (not shown in the figures) optimum transmit covariance matrices in the lower bound are almost the same as those in the upper bound for each case. In Figs. 7 and 8, we obtain similar plots for the half-duplex scenario for which capacity is obtained when $\alpha = 0$ in Fig. 7. The optimum α values for the lower and upper bounds turn out to be different in Fig. 8 for Case 3.

Next, with the source power fixed at 10 dB, we simulate the lower bound algorithm by changing the relay power. In Fig. 9, we observe that the channel is subject to the Case 2 condition when the relay power is 5–10 dB, to the Case 3 condition when the relay power is 10–15 dB, and to the Case 1 condition when the relay power is 15–30 dB. The channel saturates with relay power since, in Case 1, the relay power is large enough to forward all the information decoded at the relay node to the destination node, and the achievable rate is limited by the capacity of the source-to-relay link [4]. Having seen the effect of increasing relay power, in Fig. 10, we plot the effect of the duration that is allocated to relay silent period in half-duplex

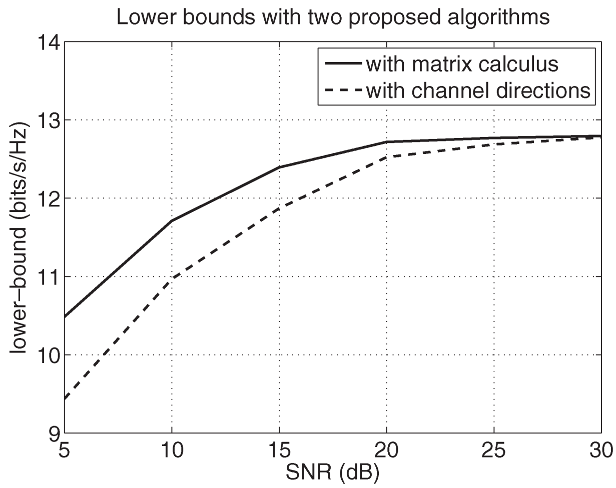

Fig. 11. Comparison of the lower bounds that are obtained by the algorithm with matrix calculus and by fixing the channel directions first and proposing an algorithm for the power values only.

transmission. We observe that the lower bound is concave in t. For the covariance matrix setting in Fig. 10, the best lower bound is obtained when $t = 0.8$.

Finally, we compare the performance of the algorithm proposed in this paper that is based on matrix calculus to an algorithm that can be derived from previous literature. In this second algorithm, transmit covariance matrices are decomposed into eigenvectors and eigenvalues. The eigenvectors are chosen cleverly, but this choice is most probably not optimal. Then, only the eigenvalues are determined using an algorithm. In Fig. 11, we clearly see that the algorithm proposed in this paper outperforms the algorithm with fixed channel directions that is proposed in [10], particularly at low SNR conditions. As SNR increases, we know in Fig. 9 that Case 1 gives the lower bound. Since Case 1 results in a single-user solution, it is no surprise that two algorithms give the same lower bound in a high-SNR scenario.

VII. Conclusion

In this paper, we have analyzed both full-duplex and half-duplex fading MIMO relay channels when the transmitters have partial CSI and the receivers have the perfect CSI. The channel capacity for such a system is not known in general. We derived DF achievable rates and cut-set upper bounds on the channel capacity, which were given in terms of max–min-type optimization problems. When the transmitters know the channel covariance information, finding the optimum source and relay transmit covariance matrices become important because power allocation over the spatial dimension of the channel has a significant impact on the performance. We use matrix differential calculus to solve the source and relay transmit covariance matrices jointly. In our method, optimum transmit covariance matrices have been found directly using a fast and efficient iterative algorithm.

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